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Matrix representation of the generators of symplectic algebras: II. The general case with explicit results for sp(6, *R*)

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Abstract. One of the important problems in Lie algebras is to determine the matrix representation of their generators in a basis associated with a given irrep of the corresponding group. For the unitary and orthogonal groups these representations were obtained by Gelfand and Zetlin. In the present paper we give results of the same degree of generality for the symplectic Lie algebras sp($2d$, R), where d is any integer, when the irreps are in the positive discrete series. The basis is constructed from powers of the raising generators acting on Gelfand states. Using the Dyson boson realisation of the generators of sp($2d$, R), and the result of Gelfand for unitary Lie algebras, the matrix representation of these generators was obtained in closed form. Explicit results are given for sp(6, R) and we sketch their application to the symplectic model of the nucleus.

1. Introduction

In the present paper we wish to extend to the symplectic algebra sp($2d$, R), where d is any integer, the methods for determining the matrix representation of their generators that were outlined in the first paper of this series for the case of sp(4, R) i.e. when $d=2$ (Castaños and Moshinsky 1987). In this introduction we shall indicate how the matrix representation can be obtained in principle, leaving for the following sections the actual procedure employed using the Dyson boson realisation of sp($2d$, R), as well as the discussion of its explicit application to sp(6, R) and to the symplectic model of the nucleus.

The set of $d(2d+1)$ generators of sp($2d$, R) will be denoted by

$$B^{\dagger}_{ij}=B^{\dagger}_{ji} \qquad C^{j}_{i} \qquad B^{ij}=B^{ji} \qquad i,j=1,2,\ldots,d \tag{1.1}$$

where in contrast to previous notation (Castaños *et al* 1982) we make use both of lower and upper indices. This would allow a discussion with different types of metric, which will be relevant in the applications to the sp(6, R) of the symplectic model of nuclear collective motions.

The commutation relations for the generators given in (1.1) are

$$[C^{j}_{i}, C^{j'}_{i'}]=C^{j'}_{i}\delta^{j}_{i'}-C^{j}_{i'}\delta^{j'}_{i} \tag{1.2a}$$

$$[B^{\dagger}_{ij}, B^{\dagger}_{i'j'}]=[B^{ij}, B^{i'j'}]=0 \tag{1.2b}$$

$$[C^{j}_{i}, B^{\dagger}_{i'j'}]=B^{\dagger}_{ij'}\delta^{j}_{i'}+B^{\dagger}_{ii'}\delta^{j}_{j'} \tag{1.2c}$$

$$[C^{j}_{i}, B^{i'j'}]=-B^{jj'}\delta^{i'}_{i}-B^{ji'}\delta^{j'}_{i} \tag{1.2d}$$

$$[B^{ij}, B^{\dagger}_{i'j'}]=C^{j}_{j'}\delta^{i}_{i'}+C^{j}_{i'}\delta^{i}_{j'}+C^{i}_{j'}\delta^{j}_{i'}+C^{i}_{i'}\delta^{j}_{j'} \tag{1.2e}$$

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and thus we see that the C_i^j, $i, j = 1, 2, \ldots, d$, are the generators of the u(d) subalgebra of sp($2d$, R) (Moshinsky 1968).

The set of generators (1.1) can be divided into three subsets of raising, weight and lowering type which are separated below by semicolons:

$$B_{ij}^{\dagger}, C_i^j \; i<j; \quad C_i^i; \quad C_i^j \; i>j, B^{ij}. \tag{1.3}$$

The lowest weight state associated with an irreducible representation of sp($2d$, R) in the positive discrete series is then characterised by a partition $[h_{1d}h_{2d} \ldots h_{dd}]$ and it can be denoted by the ket $|h_{id}\rangle$, where $i = 1, 2, \ldots, d$, which satisfies the equations

$$B_{kl}|h_{id}\rangle = 0 \qquad \forall k, l \tag{1.4a}$$

$$C_k^l|h_{id}\rangle = 0 \qquad k>l \tag{1.4b}$$

$$C_k^k|h_{id}\rangle = h_{d-k+1,d}|h_{id}\rangle \qquad k, l = 1, 2, \ldots, d. \tag{1.4c}$$

We can obtain a complete, albeit non-orthonormal, basis associated with the irrep $[h_{1d} \ldots h_{dd}]$ of sp($2d$, R) if we apply to the lowest weight state $|h_{id}\rangle$ the raising generators appearing in (1.3) before the first semicolon. We note that the application of powers of C_i^j, $i<j$ to $|h_{id}\rangle$ will produce states associated with the irrep $[h_{1d} \ldots h_{dd}]$ of U(d) (Moshinsky 1968) that would be linear combinations of the orthonormal ones introduced by Gelfand and Zetlin (Gelfand and Zetlin 1950, Moshinsky 1968). Thus in what follows we shall start not from the lowest weight state $|h_{id}\rangle$ of (1.4) but from Gelfand states characterised by irreps of the chain

$$\mathrm{U}(d) \supset \mathrm{U}(d-1) \supset \ldots \supset \mathrm{U}(j) \supset \ldots \supset \mathrm{U}(2) \supset \mathrm{U}(1) \tag{1.5}$$

which are associated with the partitions

$$[h_{1j}h_{2j} \ldots h_{jj}] \qquad j = 1, 2, \ldots, d. \tag{1.6}$$

We shall denote the corresponding states by the kets

$$|h_{ij}\rangle \qquad i \leqslant j, j = 1, 2, \ldots, d \tag{1.7}$$

where h_{ij} are non-negative integers satisfying $h_{ij} \geqslant h_{i,j-1} \geqslant h_{i+1,j}$ and, as mentioned above, these kets are orthonormal.

The set of states associated with the irrep $[h_{1d} \ldots h_{dd}]$ of sp($2d$, R) in the positive discrete series can now be constructed by the application of powers of the raising generators $B_{ij}^{\dagger}$ to the states $|h_{ij}\rangle$ of (1.7), and thus we can denote them by the ket (Deenen and Quesne 1985)

$$|N_{ij}, h_{ij}\rangle = \left(\prod_{i \leqslant j=1}^{d} (B_{ij}^{\dagger})^{N_{ij}} \right) |h_{ij}\rangle \tag{1.8}$$

where N_{ij}, $i \leqslant j = 1, \ldots, d$ are non-negative integers.

From the Hermitian property (Castaños *et al* 1982)

$$(B_{ij}^{\dagger})^{\dagger} = B^{ij} \tag{1.9}$$

and the commutation relations (1.2), it is clear that the states (1.8) are non-orthonormal but they provide a basis for the matrix representation of the generators of sp($2d$, R) as indicated in the general type of analysis of Gruber and Klimyk (1984) and, specifically in the case of sp(4, R), in the previous paper of this series (Castaños and Moshinsky 1987).

We proceed now to indicate how, in principle, one can obtain the matrix representation of the generators for a basis associated with a given irrep $[h_{1d} \ldots h_{dd}]$ of sp($2d$, R).

If we denote by X any of $d(2d+1)$ generators (1.1) of sp($2d$, R), we will try to express its action on the ket $|N_{ij}, h_{ij}\rangle$ as a linear combination of these types of states, i.e.

$$X|N_{ij}, h_{ij}\rangle = \sum_{N'_{ij}, h'_{ij}} |N'_{ij}, h'_{ij}\rangle(N'_{ij}, h'_{ij}|X|N_{ij}, h_{ij}\rangle. \tag{1.10}$$

Note that we use a round rather than an angular bracket on the left-hand side of the last term in (1.10) to emphasise that *it is not* the matrix element of X with respect to the states $|N_{ij}, h_{ij}\rangle$ but just a coefficient in the development of $X|N_{ij}, h_{ij}\rangle$ in terms of $|N'_{ij}, h'_{ij}\rangle$.

When $X = B^{\dagger}_{kl}$, $k \leq l = 1, 2, \ldots, d$, we obviously have that

$$(N_{ij} + \delta_{ik}\delta_{jl}, h_{ij}|B^{\dagger}_{kl}|N_{ij}, h_{ij}\rangle = 1 \tag{1.11}$$

while all the other coefficients in a development of the type (1.10) vanish. If we now take $X = C^l_k$ and apply it to $|N_{ij}, h_{ij}\rangle$ we obtain

$$C^l_k|N_{ij}, h_{ij}\rangle = \left[C^l_k, \prod_{i\leq j=1}^{d} (B^{\dagger}_{ij})^{N_{ij}}\right]|h_{ij}\rangle + \sum_{h'_{ij}} |N_{ij}, h'_{ij}\rangle\langle h'_{ij}|C^l_k|h_{ij}\rangle. \tag{1.12}$$

In (1.12) the $\langle h'_{ij}|C^l_k|h_{ij}\rangle$ are now the matrix elements of the generator C^l_k of the U(d) group with respect to the Gelfand states and their explicit expressions are well known (Gelfand and Zetlin 1950, Moshinsky 1968). The commutator in (1.12) can be evaluated using (1.2c) and thus it clearly gives rise to a linear combination of terms $\Pi^d_{i\leq j=1}(B^{\dagger}_{ij})^{N'_{ij}}$. In this way we can get explicitly the coefficients

$$(N'_{ij}, h'_{ij}|C^l_k|N_{ij}, h_{ij}\rangle. \tag{1.13}$$

Now turning our attention to $X = B^{kl}$ we see that

$$B^{kl}|N_{ij}, h_{ij}\rangle = \left[B^{kl}, \prod_{i\leq j=1}^{d} (B^{\dagger}_{ij})^{N_{ij}}\right]|h_{ij}\rangle \tag{1.14}$$

as the term $B^{kl}|h_{ij}\rangle = 0$. This is due to the fact that $|h_{ij}\rangle$ can be expressed as a polynomial in C^j_i, $i<j$, acting on the lowest weight state $|h_{id}\rangle$. Using the commutation relation $[C^j_i, B^{kl}]$ in (1.2d) we see that we can pass the B^{kl} through the C^j_i to have it act directly on $|h_{id}\rangle$ where it gives zero from (1.4a). For the commutator in (1.14) we can use $[B^{kl}, B^{\dagger}_{ij}]$ given by (1.2e) to introduce a $C^{j'}_{i'}$ in the monomial expression and then move the $C^{j'}_{i'}$ to the right using (1.2c) so that finally it acts on $|h_{ij}\rangle$ in the way indicated in the last term in (1.12). By this procedure we could then get explicitly the coefficients

$$(N'_{ij}, h'_{ij}|B^{kl}|N_{ij}, h_{ij}\rangle. \tag{1.15}$$

Clearly the analysis indicated, while feasible, is very cumbersome, but fortunately, as was indicated in the earlier paper (Castaños and Moshinsky 1987), it can be greatly simplified by using the Dyson boson realisation of sp($2d$, R), as we shall show in the following section.

2. The Dyson boson realisations and the matrix representation of the generators of sp($2d$, R)

In previous references (Deenen and Quesne 1984, Moshinsky 1984) it was shown that the generators of sp($2d$, R) can be expressed in terms of those of the direct sum

$$\mathrm{w}[d(d+1)/2] \oplus \mathrm{u}(d) \tag{2.1}$$

where in (2.1) w and u stand, respectively, for the Weyl and unitary Lie algebras of the dimensions indicated.

The generators of $\mathrm{w}[d(d+1)/2]$ are the creation $\beta^\dagger_{ij}$ and annihilation β^{ij} operators, with $i, j = 1, 2, \ldots, d$, satisfying the commutation relations

$$[\beta^\dagger_{ij}, \beta^\dagger_{i'j'}] = 0 \tag{2.2a}$$

$$[\beta^{ij}, \beta^{i'j'}] = 0 \tag{2.2b}$$

$$[\beta^{ij}, \beta^\dagger_{i'j'}] = \delta^i_{i'}\delta^j_{j'} + \delta^i_{j'}\delta^j_{i'} \tag{2.2c}$$

where the appearance of two sets of Kronecker deltas in (2.2c) is due to the two indices and symmetric properties of the creation and annihilation operators, i.e. $\beta^\dagger_{ij} = \beta^\dagger_{ji}$, $\beta^{ij} = \beta^{ji}$.

The generators of $\mathrm{u}(d)$ will be denoted by γ^j_i; $i, j = 1, \ldots, d$, and they satisfy the standard commutation relations for unitary Lie algebras, i.e.

$$[\gamma^j_i, \gamma^{j'}_{i'}] = \gamma^{j'}_i \delta^j_{i'} - \gamma^j_{i'} \delta^{j'}_i. \tag{2.2d}$$

Furthermore w and u are independent so that

$$[\beta^\dagger_{ij}, \gamma^{j'}_{i'}] = [\beta^{ij}, \gamma^{j'}_{i'}] = 0. \tag{2.2e}$$

The realisation of the generators of $\mathrm{sp}(2d, R)$ in terms of those of $\mathrm{w}[d(d+1)/2]$ and $\mathrm{u}(d)$ has been given in matrix notation (i.e. $\boldsymbol{B}^\dagger = \|B^\dagger_{ij}\|$, etc) by Deenen and Quesne (1984) and Moshinsky (1984). Putting the results back in components we obtain

$$B^\dagger_{ij} = \beta^\dagger_{ij} \tag{2.3a}$$

$$C^j_i = \beta^\dagger_{im}\beta^{mj} + \gamma^j_i \tag{2.3b}$$

$$B^{ij} = \beta^{im}\gamma^j_m + \beta^{jm}\gamma^i_m + \beta^{im}\beta^\dagger_{mn}\beta^{nj} - (d+1)\beta^{ij} \tag{2.3c}$$

where repeated indices m, n are summed over their values, $m, n = 1, 2, \ldots, d$.

It can be easily checked from the commutation rules (2.2) of $\beta^\dagger_{ij}, \gamma^j_i, \beta^{ij}$ that $B^\dagger_{ij}, C^j_i, B^{ij}$ defined by (2.3) satisfy the commutation rules (1.2) and, furthermore, that the right-hand side of (2.3c) is symmetric under exchange of i and j as required by $B^{ij} = B^{ji}$. We note, though, that from (2.3) and the Hermitian properties

$$(B^\dagger_{ij})^\dagger = B^{ij} \tag{2.4a}$$

$$(C^j_i)^\dagger = C^i_j \tag{2.4b}$$

of the generators of $\mathrm{sp}(2d, R)$, we conclude that

$$(\beta^\dagger_{ij})^\dagger \neq \beta^{ij} \tag{2.5}$$

and thus we are dealing with what is known as a Dyson type (Deenen and Quesne 1984, Castaños *et al* 1985, 1986) boson realisation and not a Holstein–Primakoff one.

We now define the boson states (Deenen and Quesne 1985)

$$|N_{ij}, h_{ij}\} = \prod_{i \leqslant j = 1}^{d} (\beta^\dagger_{ij})^{N_{ij}} |h_{ij}\} \tag{2.6}$$

where $|h_{ij}\}$ is the direct product of the boson vacuum and Gelfand states associated with the $\mathrm{u}(d)$ Lie algebra, so it has the properties

$$\beta^{kl}|h_{ij}\} = 0 \tag{2.7a}$$

$$\gamma^l_k |h_{ij}\} = \sum_{h'_{ij}} |h'_{ij}\}\langle h'_{ij}|\gamma^l_k|h_{ij}\rangle \tag{2.7b}$$

where the matrix elements appearing in (2.7b) are those obtained by Gelfand and Zetlin (1950), which coincide with those on the right-hand side of (1.12). We thus conclude (Castaños and Moshinsky 1987) that applying the operators on the right-hand side of (2.3) to the states (2.6) should give exactly the same result as when we apply the generators (1.1) of sp($2d, R$) to the states (1.8) characterised by the irrep $[h_{1d}h_{2d}\dots h_{dd}]$ of this Lie algebra.

We thus have an alternative procedure to the one outlined in the previous section for deriving the matrix representation of the generators sp($2d, R$) associated with a given irrep of this Lie algebra. We note, though, that in the present analysis $\beta^\dagger_{ij}$, β^{ij} are creation and annihilation operators and thus from the commutation relations (2.2c) we can interpret β^{ij} as the differential operator (Deenen and Quesne 1982, 1984)

$$\beta^{ij}=(1+\delta_{ij})\frac{\partial}{\partial\beta^\dagger_{ij}}. \tag{2.8}$$

Furthermore from (2.2e), γ^j_i will act only on the $|h_{ij}\}$ in the way indicated in (2.7b). With the help of these considerations we proceed now to apply the right-hand side of (2.3) to (2.6) to obtain the matrix representation of the generators of sp($2d, R$) that we are looking for.

We start with $B^\dagger_{kl}=\beta^\dagger_{kl}$ where $k\leqslant l$, and thus we obtain immediately

$$B^\dagger_{kl}|N_{ij}, h_{ij}\}=|N_{ij}+\delta_{ik}\delta_{jl}, h_{ij}\} \tag{2.9}$$

which coincides with the result in (1.11). Turning our attention to C^l_k we start with $k<l$ and from (2.3b) and (2.8) we write

$$\begin{aligned}C^l_k&=\sum_{m=1}^{d}\beta^\dagger_{km}(1+\delta_{ml})\frac{\partial}{\partial\beta^\dagger_{ml}}+\gamma^l_k\\ &=\sum_{m=1}^{k}\beta^\dagger_{mk}\frac{\partial}{\partial\beta^\dagger_{ml}}+\sum_{m=k+1}^{l-1}\beta^\dagger_{km}\frac{\partial}{\partial\beta^\dagger_{ml}}+2\beta^\dagger_{kl}\frac{\partial}{\partial\beta^\dagger_{ll}}+\sum_{m=l+1}^{d}\beta^\dagger_{km}\frac{\partial}{\partial\beta^\dagger_{lm}}+\gamma^l_k.\end{aligned} \tag{2.10}$$

On the right-hand side of (2.10) we decomposed the summation over m into several parts so as to be able to always write $\beta^\dagger_{ij}$ with $i\leqslant j$, both when it appears as a multiplying factor as well as when we are taking the derivative with respect to it. In this way the action of C^l_k, with $k<l$, on $|N_{ij}, h_{ij}\}$ is perfectly definite, as in the latter there only appear $\beta^\dagger_{ij}$ with $i\leqslant j$. We thus get for $k<l$ that

$$\begin{aligned}C^l_k|N_{ij}, h_{ij}\}&=\sum_{m=1}^{k}N_{ml}|N_{ij}-\delta_{im}\delta_{jl}+\delta_{im}\delta_{jk}, h_{ij}\}\\ &\quad+\sum_{m=k+1}^{l-1}N_{ml}|N_{ij}-\delta_{im}\delta_{jl}+\delta_{ik}\delta_{jm}, h_{ij}\}+2N_{ll}|N_{ij}-\delta_{il}\delta_{jl}+\delta_{ik}\delta_{jl}, h_{ij}\}\\ &\quad+\sum_{m=l+1}^{d}N_{lm}|N_{ij}-\delta_{il}\delta_{jm}+\delta_{ik}\delta_{jm}, h_{ij}\}+\sum_{h'_{ij}}|N_{ij}, h'_{ij}\}\langle h'_{ij}|\gamma^l_k|h_{ij}\rangle.\end{aligned} \tag{2.11}$$

A similar analysis when $k=l$ allows us to write

$$C^l_l=\sum_{m=1}^{l-1}\beta^\dagger_{ml}\frac{\partial}{\partial\beta^\dagger_{ml}}+2\beta^\dagger_{ll}\frac{\partial}{\partial\beta^\dagger_{ll}}+\sum_{m=l+1}^{d}\beta^\dagger_{lm}\frac{\partial}{\partial\beta^\dagger_{lm}}+\gamma^l_l \tag{2.12}$$

and thus obtain

$$C^l_l|N_{ij}, h_{ij}\}=\left(\sum_{m=1}^{l-1}N_{ml}+2N_{ll}+\sum_{m=l+1}^{d}N_{lm}+\sum_{i=1}^{l}h_{il}-\sum_{i=1}^{l-1}h_{i,l-1}\right)|N_{ij}, h_{ij}\}. \tag{2.13}$$

Finally for $k>l$ we can write

$$C^l_k=\sum_{m=1}^{l-1}\beta^\dagger_{mk}\frac{\partial}{\partial\beta^\dagger_{ml}}+2\beta^\dagger_{lk}\frac{\partial}{\partial\beta^\dagger_{ll}}+\sum_{m=l+1}^{k}\beta^\dagger_{mk}\frac{\partial}{\partial\beta^\dagger_{lm}}+\sum_{m=k+1}^{d}\beta^\dagger_{km}\frac{\partial}{\partial\beta^\dagger_{lm}}+\gamma^l_k \tag{2.14}$$

which leads to

$$\begin{aligned}C^l_k|N_{ij},h_{ij}\}&=\sum_{m=1}^{l-1}N_{ml}|N_{ij}-\delta_{im}\delta_{jl}+\delta_{im}\delta_{jk},h_{ij}\}\\&\quad+2N_{ll}|N_{ij}-\delta_{il}\delta_{jl}+\delta_{il}\delta_{jk},h_{ij}\}+\sum_{m=l+1}^{k}N_{lm}|N_{ij}-\delta_{il}\delta_{jm}+\delta_{im}\delta_{jk},h_{ij}\}\\&\quad+\sum_{m=k+1}^{d}N_{lm}|N_{ij}-\delta_{il}\delta_{jm}+\delta_{ik}\delta_{jm},h_{ij}\}+\sum_{h'_{ij}}|N_{ij},h'_{ij}\}\langle h'_{ij}|\gamma^l_k|h_{ij}\rangle.\end{aligned} \tag{2.15}$$

From the discussion (1.12) we see that (2.11), (2.13) and (2.15) provide the coefficients (1.13) required for the matrix representation of C^l_k.

We now turn our attention to B^{kl}, $k\leq l$, of (2.3c). From the commutation relations (2.2) and definition (2.3b) of C^k_l we see that it can also be written as

$$\begin{aligned}B^{kl}&=\sum_{m=1}^{d}\beta^{km}\gamma^l_m+\sum_{m=1}^{d}C^k_m\beta^{ml}\\&=\sum_{m=1}^{k-1}\gamma^l_m\frac{\partial}{\partial\beta^\dagger_{mk}}+2\gamma^l_k\frac{\partial}{\partial\beta^\dagger_{kk}}+\sum_{m=k+1}^{d}\gamma^l_m\frac{\partial}{\partial\beta^\dagger_{km}}\\&\quad+\sum_{m=1}^{l-1}C^k_m\frac{\partial}{\partial\beta^\dagger_{ml}}+C^k_l2\frac{\partial}{\partial\beta^\dagger_{ll}}+\sum_{m=l+1}^{d}C^k_m\frac{\partial}{\partial\beta^\dagger_{lm}}.\end{aligned} \tag{2.16}$$

Applying it to $|N_{ij},h_{ij}\}$ we then obtain

$$\begin{aligned}B^{kl}|N_{ij},h_{ij}\}&=\sum_{m=1}^{k-1}\sum_{h'_{ij}}N_{mk}|N_{ij}-\delta_{im}\delta_{jk},h'_{ij}\}\langle h'_{ij}|\gamma^l_m|h_{ij}\rangle\\&\quad+2\sum_{h'_{ij}}N_{kk}|N_{ij}-\delta_{ik}\delta_{jk},h'_{ij}\}\langle h'_{ij}|\gamma^l_k|h_{ij}\rangle\\&\quad+\sum_{m=k+1}^{d}\sum_{h'_{ij}}|N_{ij}-\delta_{ik}\delta_{jm},h'_{ij}\}\langle h'_{ij}|\gamma^l_m|h_{ij}\rangle\\&\quad+\sum_{m=1}^{l-1}N_{ml}C^k_m|N_{ij}-\delta_{im}\delta_{jl},h_{ij}\}+2N_{ll}C^k_l|N_{ij}-\delta_{il}\delta_{jl},h_{ij}\}\\&\quad+\sum_{m=l+1}^{d}N_{lm}C^k_m|N_{ij}-\delta_{il}\delta_{jm},h_{ij}\}.\end{aligned} \tag{2.17}$$

For the final expression we have to act with C^k_m on the kets in (2.17) which we can achieve with the help of (2.11), (2.13) or (2.15) depending on whether $m<k$, $m=k$ or $m>k$. We shall not give the explicit expressions in this general case as they become rather long and, besides, we have not given the explicit expressions for $\langle h'_{ij}|\gamma^l_k|h_{ij}\rangle$ either. We shall correct both points in the next section where we discuss the explicit matrix representation for the generators of sp(6, R).

From the analysis in (1.14) we conclude that (2.17) provides the coefficients (1.15) required for the matrix representation of B^{kl}.

It is convenient to have an independent procedure to check the elements of the matrix representation of sp(2d, R) that were derived in this section. For this purpose we introduce the second-order Casimir operator of sp(2d, R) that has the form (Castaños and Moshinsky 1986)

$$G_2=\sum_{k,l=1}^{d} B^{\dagger}_{kl}B^{kl}-\sum_{k,l=1}^{d} C^l_k C^k_l+(d+1)\sum_{k=1}^{d} C^k_k \tag{2.18}$$

where, with the help of (1.2), we easily check that G_2 commutes with all the generators of sp(2d, R). When applying G_2 to the state $|N_{ij}, h_{ij}\rangle$ of (1.8) we can then pass through all the powers of $B^{\dagger}_{ij}$, C^j_i that appear on it and have it act on the state of lowest weight $|h_{id}\rangle$ of (1.4). As from (1.2a) we can also write

$$G_2=\sum_{k,l=1}^{d} B^{\dagger}_{kl}B^{kl}-2\sum_{k<l} C^l_k C^k_l-\sum_{k>l}(C^k_k-C^l_l)-\sum_k (C^k_k)^2+(d+1)\sum_k C^k_k \tag{2.19}$$

we see from (1.4) that the effect on $|h_{id}\rangle$ of G_2, and thus also on $|N_{ij}, h_{ij}\rangle$, is given by

$$G_2|N_{ij}, h_{ij}\rangle=\Bigg(-\sum_k (h_{d-k+1,d})^2-\sum_{k>l}(h_{d-k+1,d}-h_{d-l+1,d})$$
$$+(d+1)\sum_k h_{d-k+1,d}\Bigg)|N_{ij}, h_{ij}\rangle. \tag{2.20}$$

Thus by using explicitly the action on $|N_{ij}, h_{ij}\rangle$ of each of the generators of sp(2d, R) appearing in the G_2 of (2.18), we have to arrive at the expression (2.20), and therefore can check whether any mistakes were made in deriving the matrix representation of these generators.

3. Explicit matrix representation of the generators of sp(6, R)

We have derived in the previous section the matrix representation of sp(2d, R) for any d, though some steps were indicated only symbolically, such as the matrix elements $\langle h'_{ij}|C^l_k|h_{ij}\rangle$ of the unitary Lie algebra u(d), or the application of C^k_m to the kets in (2.17). In this section we want to give explicitly all the results for $d=3$, i.e. sp(6, R), because of its interest in many physical applications. Note that sp(4, R) is also of interest but the results in this case, albeit in slightly different notation, were given in the previous paper of this series (Castaños and Moshinsky 1987).

To derive the matrix representation of sp(6, R) we need the matrix elements

$$\langle h'_{ij}|C^l_k|h_{ij}\rangle\equiv\left\langle \begin{matrix} h_{13} & h_{23} & h_{33}\\ & h'_{12} \quad h'_{22} & \\ & h'_{11} & \end{matrix}\right| C^l_k \left| \begin{matrix} h_{13} & h_{23} & h_{33}\\ & h_{12} \quad h_{22} & \\ & h_{11} & \end{matrix}\right\rangle \tag{3.1}$$

for which we shall take the values in the form presented by Moshinsky (1968). Furthermore, when useful, we shall designate our states $|N_{ij}, h_{ij}\rangle$ in the explicit form

$$|N_{ij}, h_{ij}\rangle=|N_{11}, N_{12}, N_{13}, N_{22}, N_{23}, N_{33}; h_{12}, h_{22}, h_{11}\rangle \tag{3.2}$$

suppressing the partition $[h_{13}h_{23}h_{33}]$, which designates the irrep of sp(6, R) and thus remains invariant under the application of any of the generators of this Lie algebra.

We then apply the 21 generators $B^{\dagger}_{kl}$, $k \leqslant l$; C^{l}_{k}; B^{kl}, $k \leqslant l$; $k, l = 1, 2, 3$, of sp(6, R) to the states $|N_{ij}, h_{ij}\rangle$ and from the discussion of the previous section obtain

$$B^{\dagger}_{kl}|N_{ij}, h_{ij}\rangle = |N_{ij} + \delta_{ik}\delta_{jl}, h_{ij}\rangle \qquad k \leqslant l. \tag{3.3a}$$

For C^{l}_{k} with $k < l$ we obtain

$$\begin{aligned} C^{2}_{1}|N_{ij}, h_{ij}\rangle = {} & N_{12}|N_{11}+1, N_{12}-1, N_{13}N_{22}N_{23}N_{33}, h_{ij}\rangle \\ & + 2N_{22}|N_{11}, N_{12}+1, N_{13}, N_{22}-1, N_{23}N_{33}, h_{ij}\rangle \\ & + N_{23}|N_{11}N_{12}, N_{13}+1, N_{22}, N_{23}-1, N_{33}, h_{ij}\rangle \\ & + [(h_{12}-h_{11})(h_{11}-h_{22}+1)]^{1/2}|N_{ij}; h_{12}, h_{22}, h_{11}+1\rangle \end{aligned} \tag{3.3b}$$

$$\begin{aligned} C^{3}_{1}|N_{ij}, h_{ij}\rangle = {} & N_{13}|N_{11}+1, N_{12}, N_{13}-1, N_{22}N_{23}N_{33}, h_{ij}\rangle \\ & + N_{23}|N_{11}, N_{12}+1, N_{13}N_{22}, N_{23}-1, N_{33}, h_{ij}\rangle \\ & + 2N_{33}|N_{11}N_{12}, N_{13}+1, N_{22}N_{23}, N_{33}-1, h_{ij}\rangle \\ & + \left(\frac{(h_{11}-h_{22}+1)(h_{13}-h_{12})(h_{12}-h_{23}+1)(h_{12}-h_{33}+2)}{(h_{12}-h_{22}+1)(h_{12}-h_{22}+2)}\right)^{1/2} \\ & \times |N_{ij}, h_{12}+1, h_{22}, h_{11}+1\rangle \\ & - \left(\frac{(h_{12}-h_{11})(h_{22}-h_{33}+1)(h_{23}-h_{22})(h_{13}-h_{22}+1)}{(h_{12}-h_{22})(h_{12}-h_{22}+1)}\right)^{1/2} \\ & \times |N_{ij}, h_{12}, h_{22}+1, h_{11}+1\rangle \end{aligned} \tag{3.3c}$$

$$\begin{aligned} C^{3}_{2}|N_{ij}; h_{ij}\rangle = {} & N_{13}|N_{11}, N_{12}+1, N_{13}-1, N_{22}N_{23}N_{33}, h_{ij}\rangle \\ & + N_{23}|N_{11}N_{12}N_{13}, N_{22}+1, N_{23}-1, N_{33}, h_{ij}\rangle \\ & + 2N_{33}|N_{11}N_{12}N_{13}N_{22}, N_{23}+1, N_{33}-1, h_{ij}\rangle \\ & + \left(\frac{(h_{12}-h_{11}+1)(h_{13}-h_{12})(h_{12}-h_{23}+1)(h_{12}-h_{33}+2)}{(h_{12}-h_{22}+1)(h_{12}-h_{22}+2)}\right)^{1/2} \\ & \times |N_{ij}, h_{12}+1, h_{22}, h_{11}\rangle \\ & + \left(\frac{(h_{11}-h_{22})(h_{22}-h_{33}+1)(h_{23}-h_{22})(h_{13}-h_{22}+1)}{(h_{12}-h_{22})(h_{12}-h_{22}+1)}\right)^{1/2} \\ & \times |N_{ij}, h_{12}, h_{22}+1, h_{11}\rangle. \end{aligned} \tag{3.3d}$$

For C^{k}_{k} we then have

$$C^{1}_{1}|N_{ij}, h_{ij}\rangle = (2N_{11}+N_{12}+N_{13}+h_{11})|N_{ij}, h_{ij}\rangle \tag{3.3e}$$

$$C^{2}_{2}|N_{ij}, h_{ij}\rangle = (N_{12}+2N_{22}+N_{23}+h_{12}+h_{22}-h_{11})|N_{ij}, h_{ij}\rangle \tag{3.3f}$$

$$C^{3}_{3}|N_{ij}, h_{ij}\rangle = (N_{13}+N_{23}+2N_{33}+h_{13}+h_{23}+h_{33}-h_{12}-h_{22})|N_{ij}, h_{ij}\rangle. \tag{3.3g}$$

For C^{l}_{k} with $k > l$ we obtain

$$\begin{aligned} C^{1}_{2}|N_{ij}, h_{ij}\rangle = {} & 2N_{11}|N_{11}-1, N_{12}+1, N_{13}N_{22}N_{23}N_{33}, h_{ij}\rangle \\ & + N_{12}|N_{11}, N_{12}-1, N_{13}, N_{22}+1, N_{23}N_{33}, h_{ij}\rangle \\ & + N_{13}|N_{11}N_{12}, N_{13}-1, N_{22}, N_{23}+1, N_{33}, h_{ij}\rangle \\ & + [(h_{12}-h_{11}+1)(h_{11}-h_{22})]^{1/2}|N_{ij}, h_{12}, h_{22}, h_{11}-1\rangle \end{aligned} \tag{3.3h}$$

$$\begin{aligned}C_3^1|N_{ij}, h_{ij}\rangle = {} & 2N_{11}|N_{11}-1, N_{12}, N_{13}+1, N_{22}N_{23}N_{33}, h_{ij}\rangle \\ & + N_{12}|N_{11}, N_{12}-1, N_{13}N_{22}, N_{23}+1, N_{33}, h_{ij}\rangle \\ & + N_{13}|N_{11}N_{12}, N_{13}-1, N_{22}N_{23}, N_{33}+1, h_{ij}\rangle \\ & - \left(\frac{(h_{12}-h_{11}+1)(h_{22}-h_{33})(h_{23}-h_{22}+1)(h_{13}-h_{22}+2)}{(h_{12}-h_{22}+1)(h_{12}-h_{22}+2)}\right)^{1/2} \\ & \times |N_{ij}, h_{12}, h_{22}-1, h_{11}-1\rangle \\ & + \left(\frac{(h_{11}-h_{22})(h_{13}-h_{12}+1)(h_{12}-h_{23})(h_{12}-h_{33}+1)}{(h_{12}-h_{22})(h_{12}-h_{22}+1)}\right)^{1/2} \\ & \times |N_{ij}, h_{12}-1, h_{22}, h_{11}-1\rangle \end{aligned} \tag{3.3i}$$

$$\begin{aligned}C_3^2|N_{ij}, h_{ij}\rangle = {} & N_{12}|N_{11}, N_{12}-1, N_{13}+1, N_{22}N_{23}N_{33}, h_{ij}\rangle \\ & + 2N_{22}|N_{11}N_{12}N_{13}, N_{22}-1, N_{23}+1, N_{33}, h_{ij}\rangle \\ & + N_{23}|N_{11}N_{12}N_{13}N_{22}, N_{23}-1, N_{33}+1, h_{ij}\rangle \\ & + \left(\frac{(h_{11}-h_{22}+1)(h_{22}-h_{33})(h_{23}-h_{22}+1)(h_{13}-h_{22}+2)}{(h_{12}-h_{22}+1)(h_{12}-h_{22}+2)}\right)^{1/2} \\ & \times |N_{ij}, h_{12}, h_{22}-1, h_{11}\rangle \\ & + \left(\frac{(h_{12}-h_{11})(h_{13}-h_{12}+1)(h_{12}-h_{23})(h_{12}-h_{33}+1)}{(h_{12}-h_{22})(h_{12}-h_{22}+1)}\right)^{1/2} \\ & \times |N_{ij}, h_{12}-1, h_{22}, h_{11}\rangle. \end{aligned} \tag{3.3j}$$

We now consider B^{kk} that gives

$$\begin{aligned}B^{11}|N_{ij}; h_{ij}\rangle = {} & N_{12}(N_{12}-1)|N_{11}, N_{12}-2, N_{13}, N_{22}+1, N_{23}N_{33}, h_{ij}\rangle \\ & + 2N_{12}N_{13}|N_{11}, N_{12}-1, N_{13}-1, N_{22}, N_{23}+1, N_{33}, h_{ij}\rangle \\ & + N_{13}(N_{13}-1)|N_{11}N_{12}, N_{13}-2, N_{22}N_{23}, N_{33}+1, h_{ij}\rangle \\ & + 4N_{11}(N_{11}+N_{12}+N_{13}+h_{11}-1)|N_{11}-1, N_{12}N_{13}N_{22}N_{23}N_{33}, h_{ij}\rangle \\ & + 2N_{12}[(h_{12}-h_{11}+1)(h_{11}-h_{22})]^{1/2} \\ & \times |N_{11}, N_{12}-1, N_{13}N_{22}N_{23}N_{33}, h_{12}, h_{22}, h_{11}-1\rangle \\ & - 2N_{13}\left(\frac{(h_{12}-h_{11}+1)(h_{22}-h_{33})(h_{23}-h_{22}+1)(h_{13}-h_{22}+2)}{(h_{12}-h_{22}+1)(h_{12}-h_{22}+2)}\right)^{1/2} \\ & \times |N_{11}N_{12}, N_{13}-1, N_{22}N_{23}N_{33}, h_{12}, h_{22}-1, h_{11}-1\rangle \\ & + 2N_{13}\left(\frac{(h_{11}-h_{22})(h_{13}-h_{12}+1)(h_{12}-h_{23})(h_{12}-h_{33}+1)}{(h_{12}-h_{22})(h_{12}-h_{22}+1)}\right)^{1/2} \\ & \times |N_{11}N_{12}, N_{13}-1, N_{22}N_{23}N_{33}, h_{12}-1, h_{22}, h_{11}-1\rangle \end{aligned} \tag{3.3k}$$

$$\begin{aligned}
B^{22}|N_{ij}, h_{ij}\rangle = {} & N_{12}(N_{12}-1)|N_{11}+1, N_{12}-2, N_{13}N_{22}N_{23}N_{33}, h_{ij}\rangle \\
& +2N_{12}N_{23}|N_{11}, N_{12}-1, N_{13}+1, N_{22}, N_{23}-1, N_{33}, h_{ij}\rangle \\
& +N_{23}(N_{23}-1)|N_{11}N_{12}N_{13}N_{22}, N_{23}-2, N_{33}+1, h_{ij}\rangle \\
& +4N_{22}(N_{12}+N_{22}+N_{23}+h_{12}+h_{22}-h_{11}-1) \\
& \times|N_{11}N_{12}N_{13}, N_{22}-1, N_{23}N_{33}, h_{ij}\rangle+2N_{12}[(h_{12}-h_{11})(h_{11}-h_{22}+1)]^{1/2} \\
& \times|N_{11}, N_{12}-1, N_{13}N_{22}N_{23}N_{33}, h_{12}, h_{22}, h_{11}+1\rangle \\
& +2N_{23}\left(\frac{(h_{11}-h_{22}+1)(h_{22}-h_{33})(h_{23}-h_{22}+1)(h_{13}-h_{22}+2)}{(h_{12}-h_{22}+1)(h_{12}-h_{22}+2)}\right)^{1/2} \\
& \times|N_{11}N_{12}N_{13}N_{22}, N_{23}-1, N_{33}, h_{12}, h_{22}-1, h_{11}\rangle \\
& +2N_{23}\left(\frac{(h_{12}-h_{11})(h_{13}-h_{12}+1)(h_{12}-h_{23})(h_{12}-h_{33}+1)}{(h_{12}-h_{22})(h_{12}-h_{22}+1)}\right)^{1/2} \\
& \times|N_{11}N_{12}N_{13}N_{22}, N_{23}-1, N_{33}, h_{12}-1, h_{22}, h_{11}\rangle
\end{aligned} \tag{3.3l}$$

$$\begin{aligned}
B^{33}|N_{ij}, h_{ij}\rangle = {} & N_{13}(N_{13}-1)|N_{11}+1, N_{12}, N_{13}-2, N_{22}N_{23}N_{33}, h_{ij}\rangle \\
& +2N_{13}N_{23}|N_{11}, N_{12}+1, N_{13}-1, N_{22}, N_{23}-1, N_{33}, h_{ij}\rangle \\
& +N_{23}(N_{23}-1)|N_{11}N_{12}N_{13}, N_{22}+1, N_{23}-2, N_{33}, h_{ij}\rangle \\
& +4N_{33}(N_{13}+N_{23}+N_{33}+h_{13}+h_{23}+h_{33}-h_{12}-h_{22}-1) \\
& \times|N_{11}N_{12}N_{13}N_{22}N_{23}, N_{33}-1, h_{ij}\rangle \\
& +2N_{13}\left(\frac{(h_{11}-h_{22}+1)(h_{13}-h_{12})(h_{12}-h_{23}+1)(h_{12}-h_{33}+2)}{(h_{11}-h_{22}+1)(h_{12}-h_{22}+2)}\right)^{1/2} \\
& \times|N_{11}N_{12}, N_{13}-1, N_{22}N_{23}N_{33}, h_{12}+1, h_{22}, h_{11}+1\rangle \\
& -2N_{13}\left(\frac{(h_{12}-h_{11})(h_{22}-h_{33}+1)(h_{23}-h_{22})(h_{13}-h_{22}+1)}{(h_{12}-h_{22})(h_{12}-h_{22}+1)}\right)^{1/2} \\
& \times|N_{11}N_{12}, N_{13}-1, N_{22}N_{23}N_{33}, h_{12}, h_{22}+1, h_{11}+1\rangle \\
& +2N_{23}\left(\frac{(h_{12}-h_{11}+1)(h_{13}-h_{12})(h_{12}-h_{23}+1)(h_{12}-h_{33}+2)}{(h_{12}-h_{22}+1)(h_{12}-h_{22}+2)}\right)^{1/2} \\
& \times|N_{11}N_{12}N_{13}N_{22}, N_{23}-1, N_{33}; h_{12}+1, h_{22}, h_{11}\rangle \\
& +2N_{23}\left(\frac{(h_{11}-h_{22})(h_{22}-h_{33}+1)(h_{23}-h_{22})(h_{13}-h_{22}+1)}{(h_{12}-h_{22})(h_{12}-h_{22}+1)}\right)^{1/2} \\
& \times|N_{11}N_{12}N_{13}N_{22}, N_{23}-1, N_{33}, h_{12}, h_{22}+1, h_{11}\rangle.
\end{aligned} \tag{3.3m}$$

Finally for B^{kl} with $k<l$ we obtain

$$
\begin{aligned}
B^{12}|N_{ij},h_{ij}\rangle = {} & 2N_{11}h_{11}|N_{11}-1, N_{12}N_{13}N_{22}N_{23}N_{33}, h_{ij}\rangle \\
& + N_{12}(2N_{11}+N_{12}+N_{13}+2N_{22}+N_{23}+h_{12}+h_{22}-1) \\
& \times|N_{11}, N_{12}-1, N_{13}N_{22}N_{23}N_{33}, h_{ij}\rangle \\
& +4N_{11}N_{22}|N_{11}-1, N_{12}+1, N_{13}, N_{22}-1, N_{23}N_{33}; h_{ij}\rangle \\
& +2N_{22}N_{13}|N_{11}N_{12}, N_{13}-1, N_{22}-1, N_{23}+1, N_{33}, h_{ij}\rangle \\
& +2N_{11}N_{23}|N_{11}-1, N_{12}, N_{13}+1, N_{22}, N_{23}-1, N_{33}, h_{ij}\rangle \\
& + N_{13}N_{23}|N_{11}N_{12}, N_{13}-1, N_{22}, N_{23}-1, N_{33}+1, h_{ij}\rangle \\
& +2N_{22}[(h_{12}-h_{11}+1)(h_{11}-h_{22})]^{1/2} \\
& \times|N_{11}N_{12}N_{13}, N_{22}-1, N_{23}N_{33}, h_{12}, h_{22}, h_{11}-1\rangle \\
& + N_{13}\left(\frac{(h_{11}-h_{22}+1)(h_{22}-h_{33})(h_{23}-h_{22}+1)(h_{13}-h_{22}+2)}{(h_{12}-h_{22}+1)(h_{12}-h_{22}+2)}\right)^{1/2} \\
& \times|N_{11}N_{12}, N_{13}-1, N_{22}N_{23}N_{33}, h_{12}, h_{22}-1, h_{11}\rangle \\
& + N_{13}\left(\frac{(h_{12}-h_{11})(h_{13}-h_{12}+1)(h_{12}-h_{23})(h_{12}-h_{33}+1)}{(h_{12}-h_{22})(h_{12}-h_{22}+1)}\right)^{1/2} \\
& \times|N_{11}N_{12}, N_{13}-1, N_{22}N_{23}N_{33}, h_{12}-1, h_{22}, h_{11}\rangle \\
& - N_{23}\left(\frac{(h_{12}-h_{11}+1)(h_{22}-h_{33})(h_{23}-h_{22}+1)(h_{13}-h_{22}+2)}{(h_{12}-h_{22}+1)(h_{12}-h_{22}+2)}\right)^{1/2} \\
& \times|N_{11}N_{12}N_{13}N_{22}, N_{23}-1, N_{33}, h_{12}, h_{22}-1, h_{11}-1\rangle \\
& + N_{23}\left(\frac{(h_{11}-h_{22})(h_{13}-h_{12}+1)(h_{12}-h_{23})(h_{12}-h_{33}+1)}{(h_{12}-h_{22})(h_{12}-h_{22}+1)}\right)^{1/2} \\
& \times|N_{11}N_{12}N_{13}N_{22}, N_{23}-1, N_{33}, h_{12}-1, h_{22}, h_{11}-1\rangle \qquad (3.3n)
\end{aligned}
$$

$$
\begin{aligned}
B^{13}|N_{ij},h_{ij}\rangle = {} & N_{13}(2N_{11}+N_{12}+N_{13}+N_{23}+2N_{33}+h_{13}+h_{23}+h_{33}-h_{12}+h_{11}-h_{22}-1) \\
& \times|N_{11}N_{12}, N_{13}-1, N_{22}N_{23}N_{33}, h_{ij}\rangle \\
& +2N_{11}N_{23}|N_{11}-1, N_{12}+1, N_{13}N_{22}, N_{23}-1, N_{33}, h_{ij}\rangle \\
& + N_{12}N_{23}|N_{11}, N_{12}-1, N_{13}, N_{22}+1, N_{23}-1, N_{33}, h_{ij}\rangle \\
& +4N_{11}N_{33}|N_{11}-1, N_{12}, N_{13}+1, N_{22}N_{23}, N_{33}-1, h_{ij}\rangle \\
& +2N_{12}N_{33}|N_{11}, N_{12}-1, N_{13}N_{22}, N_{23}+1, N_{33}-1, h_{ij}\rangle \\
& + N_{23}[(h_{12}-h_{11}+1)(h_{11}-h_{22})]^{1/2} \\
& \times|N_{11}N_{12}N_{13}N_{22}, N_{23}-1, N_{33}, h_{12}, h_{22}, h_{11}-1\rangle \\
& +2N_{11}\left(\frac{(h_{11}-h_{22}+1)(h_{13}-h_{12})(h_{12}-h_{23}+1)(h_{12}-h_{33}+2)}{(h_{12}-h_{22}+1)(h_{12}-h_{22}+2)}\right)^{1/2} \\
& \times|N_{11}-1, N_{12}N_{13}N_{22}N_{23}N_{33}, h_{12}+1, h_{22}, h_{11}+1\rangle \\
& -2N_{11}\left(\frac{(h_{12}-h_{11})(h_{22}-h_{33}+1)(h_{23}-h_{22})(h_{13}-h_{22}+1)}{(h_{12}-h_{22})(h_{12}-h_{22}+1)}\right)^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&\times|N_{11}-1, N_{12}N_{13}N_{22}N_{23}N_{33}, h_{12}, h_{22}+1, h_{11}+1\rangle \\
&+N_{12}\left(\frac{(h_{12}-h_{11}+1)(h_{13}-h_{12})(h_{12}-h_{23}+1)(h_{12}-h_{33}+2)}{(h_{12}-h_{22}+1)(h_{12}-h_{22}+2)}\right)^{1/2} \\
&\times|N_{11}, N_{12}-1, N_{13}N_{22}N_{23}N_{33}, h_{12}+1, h_{22}, h_{11}\rangle \\
&+N_{12}\left(\frac{(h_{11}-h_{22})(h_{22}-h_{33}+1)(h_{23}-h_{22})(h_{13}-h_{22}+1)}{(h_{12}-h_{22})(h_{12}-h_{22}+1)}\right)^{1/2} \\
&\times|N_{11}, N_{12}-1, N_{13}N_{22}N_{23}N_{33}, h_{12}, h_{22}+1, h_{11}\rangle \\
&-2N_{33}\left(\frac{(h_{12}-h_{11}+1)(h_{22}-h_{33})(h_{23}-h_{22}+1)(h_{13}-h_{22}+2)}{(h_{12}-h_{22}+1)(h_{12}-h_{22}+2)}\right)^{1/2} \\
&\times|N_{11}N_{12}N_{13}N_{22}N_{23}, N_{33}-1, h_{12}, h_{22}-1, h_{11}-1\rangle \\
&+2N_{33}\left(\frac{(h_{11}-h_{22})(h_{13}-h_{12}+1)(h_{12}-h_{23})(h_{12}-h_{33}+1)}{(h_{12}-h_{22})(h_{12}-h_{22}+1)}\right)^{1/2} \\
&\times|N_{11}N_{12}N_{13}N_{22}N_{23}, N_{33}-1, h_{12}-1, h_{22}, h_{11}-1\rangle
\end{aligned}
\tag{3.3o}
$$

$$
\begin{aligned}
B^{23}|N_{ij}, h_{ij}\rangle = {} & N_{23}(N_{12}+N_{13}+2N_{22}+N_{23}+2N_{33}+h_{13}+h_{23}+h_{33}-h_{11}-1) \\
&\times|N_{11}N_{12}N_{13}N_{22}, N_{23}-1, N_{33}, h_{ij}\rangle \\
&+N_{12}N_{13}|N_{11}+1, N_{12}-1, N_{13}-1, N_{22}N_{23}N_{33}, h_{ij}\rangle \\
&+2N_{22}N_{13}|N_{11}, N_{12}+1, N_{13}-1, N_{22}-1, N_{23}N_{33}, h_{ij}\rangle \\
&+2N_{12}N_{33}|N_{11}, N_{12}-1, N_{13}+1, N_{22}N_{23}, N_{33}-1, h_{ij}\rangle \\
&+4N_{22}N_{33}|N_{11}N_{12}N_{13}, N_{22}-1, N_{23}+1, N_{33}-1, h_{ij}\rangle \\
&+N_{13}[(h_{12}-h_{11})(h_{11}-h_{22}+1)]^{1/2} \\
&\times|N_{11}N_{12}, N_{13}-1, N_{22}N_{23}N_{33}, h_{12}, h_{22}, h_{11}+1\rangle \\
&+N_{12}\left(\frac{(h_{11}-h_{22}+1)(h_{13}-h_{12})(h_{12}-h_{23}+1)(h_{12}-h_{33}+2)}{(h_{12}-h_{22}+1)(h_{12}-h_{22}+2)}\right)^{1/2} \\
&\times|N_{11}, N_{12}-1, N_{13}N_{22}N_{23}N_{33}, h_{12}+1, h_{22}, h_{11}+1\rangle \\
&-N_{12}\left(\frac{(h_{12}-h_{11})(h_{22}-h_{33}+1)(h_{23}-h_{22})(h_{13}-h_{22}+1)}{(h_{12}-h_{22})(h_{12}-h_{22}+1)}\right)^{1/2} \\
&\times|N_{11}, N_{12}-1, N_{13}N_{22}N_{23}N_{33}, h_{12}, h_{22}+1, h_{11}+1\rangle \\
&+2N_{22}\left(\frac{(h_{12}-h_{11}+1)(h_{13}-h_{12})(h_{12}-h_{23}+1)(h_{12}-h_{33}+2)}{(h_{12}-h_{22}+1)(h_{12}-h_{22}+2)}\right)^{1/2} \\
&\times|N_{11}N_{12}N_{13}, N_{22}-1, N_{23}N_{33}, h_{12}+1, h_{22}, h_{11}\rangle \\
&+2N_{22}\left(\frac{(h_{11}-h_{22})(h_{22}-h_{33}+1)(h_{23}-h_{22})(h_{13}-h_{22}+1)}{(h_{12}-h_{22})(h_{12}-h_{22}+1)}\right)^{1/2} \\
&\times|N_{11}N_{12}N_{13}, N_{22}-1, N_{23}N_{33}, h_{12}, h_{22}+1, h_{11}\rangle \\
&+2N_{33}\left(\frac{(h_{11}-h_{22}+1)(h_{22}-h_{33})(h_{23}-h_{22}+1)(h_{13}-h_{22}+2)}{(h_{12}-h_{22}+1)(h_{12}-h_{22}+2)}\right)^{1/2} \\
&\times|N_{11}N_{12}N_{13}N_{22}N_{23}, N_{33}-1, h_{12}, h_{22}-1, h_{11}\rangle \\
&+2N_{33}\left(\frac{(h_{12}-h_{11})(h_{13}-h_{12}+1)(h_{12}-h_{23})(h_{12}-h_{33}+1)}{(h_{12}-h_{22})(h_{12}-h_{22}+1)}\right)^{1/2} \\
&\times|N_{11}N_{12}N_{13}N_{22}N_{23}, N_{33}-1, h_{12}-1, h_{22}, h_{11}\rangle.
\end{aligned}
\tag{3.3p}
$$

From (3.3) we then obtain the coefficients (1.11), (1.13) and (1.15) in the matrix representation of the generators $B^{\dagger}_{kl}$, C^{l}_{k}, B^{kl} of sp(6, R). In the next section we apply these results for the determination of the matrix representation of Hamiltonians in the enveloping algebra of sp(6, R).

4. Applications to the symplectic model of the nucleus

In the symplectic model of the nucleus (Rosensteel and Rowe 1976, 1977, 1980, Filippov *et al* 1973, 1981, Vanagas and Kalinauskas 1973, Vanagas *et al* 1975, Vanagas 1977, 1980, Weaver *et al* 1976, Castaños *et al* 1982, Moshinsky 1984, Suzuki and Hecht 1986) the Hamiltonian H is in the enveloping algebra of sp(6, R) and thus can be expressed as a polynomial in the generators $B^{\dagger}_{kl}$, C^{l}_{k}, B^{kl}, $k, l = 1, 2, 3$, that is Hermitian and invariant under rotation and time reflections (Chacón *et al* 1987). In the discussion of these Hamiltonians we analyse first the appearance of the orthogonal group O(3) in the picture and then the procedure for getting energy levels when the matrix representation of H is with respect to a non-orthonormal set of states such as $|N_{ij}, h_{ij}\rangle$.

4.1. *The use of spherical components*

So far in sp(6, R) we used the indices i, j or k, l that take the values $1, 2, 3$. From the beginning we considered a notation that would allow us to interpret these indices in any kind of metric. In what follows we shall use a metric associated with spherical components that take the values $1, 0, -1$ related to $1, 2, 3$ by the correspondence

$$\begin{array}{ccc} 1 & 0 & -1 \\ \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 \end{array} . \tag{4.1}$$

When we are speaking of the generators of sp(6, R) with the notation $q = 1, 0, -1$ for the indices, we shall denote them with a bar above, while when we put them in the notation $i = 1, 2, 3$ we shall express them as before. The raising and lowering of the indices in spherical component notation is the standard one, e.g.

$$\bar{C}^{q}_{r} = (-1)^{q}\bar{C}_{r,-q} = (-1)^{r}\bar{C}^{-r,q} \qquad q, r = 1, 0, -1. \tag{4.2}$$

The correspondence in the barred and unbarred notation is given by (4.1) and thus, for example,

$$\bar{C}^{0}_{1} = C^{2}_{1}. \tag{4.3}$$

The spherical components of angular momentum in terms of the generators $\bar{C}^{r}_{q}$ of u(3) are given by (Moshinsky 1968)

$$L_{1} = -(\bar{C}^{0}_{1} + \bar{C}^{-1}_{0}) = -(C^{2}_{1} + C^{3}_{2}) \tag{4.4a}$$

$$L_{0} = \bar{C}^{1}_{1} - \bar{C}^{-1}_{-1} = C^{1}_{1} - C^{3}_{3} \tag{4.4b}$$

$$L_{-1} = (\bar{C}^{1}_{0} + \bar{C}^{0}_{-1}) = (C^{1}_{2} + C^{2}_{3}) \tag{4.4c}$$

with the Casimir operator of O(3) being

$$L^{2} = -(L_{1}L_{-1} + L_{-1}L_{1}) + L^{2}_{0}. \tag{4.5}$$

We note from (4.4*b*) that the states $|N_{ij}, h_{ij}\rangle$ correspond to a definite eigenvalue M of the angular momentum projection L_0, which from (3.3*e*, *g*) is given by

$$M = (2N_{11} + N_{12} - N_{23} - 2N_{33} + h_{11} + h_{12} + h_{22} - h_{13} - h_{23} - h_{33}). \tag{4.6}$$

On the other hand the states $|N_{ij}, h_{ij}\rangle$ are not eigenstates of the total angular momentum L^2, though one can find linear combinations of them with this property, as will be outlined below.

Note that the states $|N_{ij}, h_{ij}\rangle$ are also eigenstates of the operator

$$\mathcal{N} \equiv \tfrac{1}{2}(\bar{C}_1^1 + \bar{C}_0^0 + \bar{C}_{-1}^{-1}) = \tfrac{1}{2}(C_1^1 + C_2^2 + C_3^3) \tag{4.7}$$

with eigenvalues

$$N = \sum_{i \leqslant j = 1}^{3} N_{ij} + \tfrac{1}{2}(h_{13} + h_{23} + h_{33}). \tag{4.8}$$

The Hamiltonians H in the enveloping algebra of sp(6, R) will commute then with the components of the angular momentum, i.e.

$$[L_q, H] = 0 \qquad q = 1, 0, -1 \tag{4.9}$$

and besides they are Hermitian and invariant under time reflection.

4.2. *Energy levels for a non-orthonormal basis*

From (3.3) we can obtain the representation of H in the basis $|N_{ij}, h_{ij}\rangle$, i.e.

$$H|N_{ij}, h_{ij}\rangle = \sum_{N'_{ij}, h'_{ij}} |N'_{ij}, h'_{ij}\rangle (N'_{ij}, h'_{ij}|H|N_{ij}, h_{ij}\rangle \tag{4.10}$$

and we shall proceed to show that the energy levels for this Hamiltonian are given by the secular equation (Moshinsky and Seligman 1971)

$$\det \left\| (N'_{ij}, h'_{ij}|H|N_{ij}, h_{ij}\rangle - E \prod_{i \leqslant j} \delta_{N'_{ij}, N_{ij}} \delta_{h'_{ij}, h_{ij}} \right\| = 0. \tag{4.11}$$

As the states $|N_{ij}, h_{ij}\rangle$ are non-orthonormal we first show that the coefficients appearing on the right-hand side of (4.10) are matrix elements between the kets $|N_{ij}, h_{ij}\rangle$ and dual ones $|N'_{ij}, h'_{ij})$, to be defined below, that are orthonormal to them. We then will see that we will get the same secular equation (4.11) if we had started with a complete orthonormalised set of states from the very beginning, which we could denote by the square bracket ket $|N_{ij}, h_{ij}]$.

To simplify the discussion of the steps mentioned in the previous paragraph let us designate our states by the single quantum number ν taking the values $\nu = 1, 2, \ldots, n$ so that (4.10) can be rewritten

$$H|\nu\rangle = \sum_{\nu'} |\nu'\rangle (\nu'|H|\nu\rangle. \tag{4.12}$$

The matrix of scalar products

$$\|\langle \nu'|\nu\rangle\| \tag{4.13}$$

which is clearly Hermitian, i.e. $\langle \nu'|\nu\rangle = \langle \nu|\nu'\rangle^*$, can be inverted to give a matrix we denote by

$$\|(\nu'|\nu)\|. \tag{4.14}$$

We now introduce a dual state which we denote by the round ket $|\nu')$ and define it by

$$|\nu') = \sum_{\nu''} |\nu'')(\nu''|\nu'). \tag{4.15}$$

Clearly then

$$\begin{aligned}(\nu'|\nu\rangle &= \sum_{\nu''} (\nu''|\nu')^* \langle \nu''|\nu\rangle \\ &= \left(\sum_{\nu''} \langle \nu|\nu''\rangle (\nu''|\nu')\right)^* = \delta_{\nu\nu'}\end{aligned} \tag{4.16}$$

where we used the Hermitian property of $\langle\nu''|\nu\rangle$ and the reciprocal relation between the matrices (4.13) and (4.14). We see then that the coefficients appearing on the right-hand side of (4.12) are matrix elements of H between an angular ket $|\nu\rangle$ and a dual round bra $(\nu'|$ of this non-orthonormal system of states.

We now turn our attention to the orthonormal set of states $|\nu]$ that we can build from the angular kets $|\nu\rangle$. As $\|\langle\nu'|\nu\rangle\|$ is Hermitian there is a unitary matrix $\|U_{\nu'\nu}\|$ that diagonalises it and all the eigenvalues are real and positive so we can designate them by λ_ν^2, $\nu = 1, 2, \ldots, n$. We now define

$$|\bar\nu] = \sum_{\nu} |\nu\rangle U_{\nu\bar\nu} \lambda_{\bar\nu}^{-1} \tag{4.17}$$

from which we have

$$[\bar\nu'|\bar\nu] = \lambda_{\bar\nu}^{-1}\lambda_{\bar\nu'}^{-1} \sum_{\nu'\nu} U^{\dagger}_{\bar\nu'\nu'} \langle\nu'|\nu\rangle U_{\nu\bar\nu} = \delta_{\bar\nu'\bar\nu}. \tag{4.18}$$

The square bracket kets $|\nu]$ are then orthonormal and if we have the development

$$H|\bar\nu] = \sum_{\bar\nu'} |\bar\nu'][\bar\nu'|H|\bar\nu] \tag{4.19}$$

then the energy levels are clearly given by the secular equation

$$\det\|[\bar\nu'|H|\bar\nu] - E\delta_{\bar\nu'\bar\nu}\| = 0. \tag{4.20}$$

We note, though, that by inverting equation (4.17) we obtain

$$|\nu\rangle = \sum_{\bar\nu} U^{\dagger}_{\bar\nu\nu} \lambda_{\bar\nu} |\bar\nu] \tag{4.21}$$

while a similar analysis using $\|(\nu'|\nu)\|$ gives

$$|\nu') = \sum_{\bar\nu'} U^{\dagger}_{\bar\nu'\nu'} \lambda_{\bar\nu'}^{-1} |\bar\nu'] \tag{4.22}$$

which can be checked by the fact that we have $(\nu'|\nu\rangle = \delta_{\nu'\nu}$. We see then immediately that the secular equation

$$\det\|(\nu'|H|\nu\rangle - E\delta_{\nu'\nu}\| = 0 \tag{4.23}$$

gives the same values of E as (4.20), as from (4.21) and (4.22) the matrices appearing in (4.20) and (4.23) are related by a similarity transformation.

In our argument we considered only a finite number of states, while $|N_{ij}, h_{ij}\rangle$ has an infinite number. We note though that the scalar products $\langle N'_{ij}, h'_{ij}|N_{ij}, h_{ij}\rangle$ vanish if they have different values M, N given in (4.6) and (4.8) in bra and ket and thus our discussion can be limited to subsets of $|N_{ij}, h_{ij}\rangle$ with fixed N, M which from (4.6) and (4.8) are finite in number.

The analysis given above then justifies our assertion that the energy levels are given by the secular equation (4.11) in which we have the matrix representation

$\langle N'_{ij}, h'_{ij} | H | N_{ij}, h_{ij}\rangle$ of the Hamiltonian H. The eigenstates of H can be obtained as linear combinations of $| N_{ij}, h_{ij}\rangle$ in the process of diagonalising the matrix representation of H. Note incidentally that the overlaps of the states $| N_{ij}, h_{ij}\rangle$ can be written as

$$\langle N'_{ij}, h'_{ij} | N_{ij}, h_{ij}\rangle = \left\langle h'_{ij} \left| \left(\prod_{i\leq j=1}^{3} (B^{ij})^{N'_{ij}} \right) \right| N_{ij}, h_{ij} \right\rangle \tag{4.24}$$

and thus using the matrix representation of B^{ij} given by (3.3k)-(3.3p) we can find a recursion relation for evaluating these overlaps. This would allow us to use the eigenstates of H in the process of evaluating the expectation values of other operators such as the angular momentum L^2 as well as those that determine the shape (Chacón *et al* 1986).

4.3. *Hamiltonians in the enveloping algebra of sp(6, R) up to second degree in the generators*

If we look at the 21 generators of sp(6, R) we clearly see that those that are invariant under rotation are

$$\sum_{q=-1}^{1} \bar{C}_q^q = \sum_{i=1}^{3} C_i^i = 2\mathcal{N} \tag{4.25a}$$

$$\sum_{q=-1}^{1} (-1)^q \bar{B}^\dagger_{q,-q} = -2B^\dagger_{13} + B^\dagger_{22} \equiv 2B^\dagger \tag{4.25b}$$

$$\sum_{q=-1}^{1} (-1)^q \bar{B}^{q,-q} = -2B^{13} + B^{22} \equiv 2B. \tag{4.25c}$$

If we require further that they should be Hermitian and invariant under time reflection (where the latter allows only *real* linear combinations or polynomials in the generators (Chacón *et al* 1986)) we obtain

$$\mathcal{N},\ B^\dagger + B. \tag{4.26}$$

Note that $\mathcal{N}$, B^+, B close under commutation, i.e.

$$[\mathcal{N}, B^\dagger] = B^\dagger \tag{4.27a}$$

$$[\mathcal{N}, B] = -B \tag{4.27b}$$

$$[B, B^\dagger] = 2\mathcal{N} \tag{4.27c}$$

and thus are a sp(2, R) subalgebra of sp(6, R).

Now turning our attention to the polynomials of second degree in the generators we see that those that are invariant under rotation are

$$\sum_{q,q'=-1}^{1} \bar{C}_q^{q'} \bar{C}_{q'}^q = \sum_{i,j=1}^{3} C_i^j C_j^i \equiv \Gamma \tag{4.28a}$$

$$\sum_{q,q'} (-1)^q \bar{B}^\dagger_{q',-q} \bar{C}_q^{q'} = \sum_{i=1}^{3} \sum_{j=1}^{3} [(-1)^j B^\dagger_{i,4-j} C_j^i] \equiv A^\dagger \tag{4.28b}$$

$$\sum_{q,q'} (-1)^q \bar{C}_{q'}^q \bar{B}^{-q,q'} = \sum_{i=1}^{3} \sum_{j=1}^{3} [(-1)^j C_i^j B^{i,4-j}] \equiv A \tag{4.28c}$$

$$\sum_{q,q'} (-1)^{q+q'} \bar{B}^\dagger_{qq'} \bar{B}^\dagger_{-q,-q'} = \sum_{i,j} (-1)^{i+j} B^\dagger_{ij} B^\dagger_{4-i,4-j} \equiv \Delta^\dagger \tag{4.28d}$$

$$\sum_{q,q'} (-1)^{q+q'} \bar{B}^{qq'} \bar{B}^{-q,-q'} = \sum_{i,j} (-1)^{i+j} B^{ij} B^{4-i,4-j} \equiv \Delta \tag{4.28e}$$

$$\sum_{q,q'} \bar{B}^\dagger_{qq'} \bar{B}^{qq'} = \sum_{i,j} B^\dagger_{ij} B^{ij} \equiv K \tag{4.28f}$$

to which we have to add L^2 of (4.5) and the squares and products of those appearing in (4.25). The Hermitian property and invariance under time reflection gives us the following possibilities for terms in H of second degree in the generators:

$$\Gamma, A^{\dagger}+A, \Delta^{\dagger}+\Delta, K, B^{\dagger}B, B^{\dagger 2}+B^2, (B^{\dagger}+B)\mathcal{N}+\mathcal{N}(B^{\dagger}+B), \mathcal{N}^2, L^2. \tag{4.29}$$

We note that among those appearing in (4.29) are Casimir operators of subalgebras of sp(6, R) such as

$$\Gamma, \quad L^2, \quad B^{\dagger}B-\mathcal{N}(\mathcal{N}-1) \tag{4.30}$$

which are, respectively, those of u(3), o(3) and sp(2, R). No similar identification was found for the other operators in (4.29) as Casimir operators of other subalgebras of sp(6, R) such as, for example cm(3), are of third and fourth degree in the generators (Weaver *et al* 1976).

The matrix representation of the operators (4.26) and (4.29) can be found immediately from (3.3) and thus the energy levels associated with a linear combination of them or of terms involving even higher powers of the generators, can be obtained from the secular equation (4.11).

Other methods for obtaining computer programs for the generators of sp(6, R) and the energy levels of Hamiltonians in the enveloping algebra have been given by Rowe (1985).

5. Conclusion

We have given in this paper the matrix representation of the generators of sp($2d$, R) with respect to the states $|N_{ij}, h_{ij}\rangle$ characterised by the irrep $[h_{1d}, \ldots, h_{dd}]$ in the positive discrete series of this Lie algebra. The explicit form of these results for the case of sp(6, R) allows us to consider the matrix representation of Hamiltonians in the enveloping algebra, and from it the energy levels of these Hamiltonians as well as the eigenstates expressed as linear combinations of the states $|N_{ij}, h_{ij}\rangle$. Applications of these procedures to specific nuclei and interactions will be considered in other publications.

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